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Profinite Completion and Generalizations of a Theorem of Blackburn

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INTRODUCTION

The theorem of Blackburn referred to in the title is the following [1, 2]:

BLACKBURN'S THEOREM. *Let G be a finitely generated nilpotent group. Then two elements of G are conjugate if and only if their images in every finite quotient group of G are conjugate.*

We generalize this theorem in two, related directions. First we consider a finitely generated nilpotent group Q operating nilpotently on a finitely generated nilpotent group N and prove that two elements of N are in the same Q-orbit if and only if their images in every finite quotient Q-group of N are in the same Q-orbit; this is the equivalence (A) $\Leftrightarrow$ (C) of Theorem 3.1. Our method of proof is to proceed via the concept of the action of the profinite completion, $\hat{Q}$, of Q on the profinite completion, $\hat{N}$, of N, induced by the given action of Q on N—a concept we introduce in Section 2. Since N is residually finite, the completion map $c: N \to \hat{N}$ embeds N in $\hat{N}$, and we show that the induced map of orbit sets $N \mid Q \to \hat{N} \mid \hat{Q}$ is also injective; this is the equivalence (A) $\Leftrightarrow$ (B) of Theorem 3.1. We remark that the equivalence (B) $\Leftrightarrow$ (C) is fairly routine in the original setting of Blackburn's theorem,[1] but presents a somewhat more complex aspect in our generalization.

Our second generalization is of a homotopy-theoretic nature. We consider a nilpotent space X of finite type and its profinite completion $c: X \to \hat{X}$. We then prove that if W is a homologically finite CW-complex, then the induced mapping of *free* homotopy sets $c_*: (W, X) \to (W, \hat{X})$ is injective. It was, of course, already known [3, 4] that the induced mapping of *based* homotopy sets is

[1] Thus, we may regard Blackburn's theorem as asserting that two elements a, b of G are conjugate in G if and only if they are conjugate in $\hat{G}$.

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injective if W is connected. If we take $W = S^1$, $X = K(G, 1)$, with G finitely generated nilpotent, then the latter result on based homotopy generalizes the fact that $c: G \to \hat{G}$ is injective, while the former result essentially generalizes Blackburn's theorem. The fact that $c_*: (W, X) \to (W, \hat{X})$ is injective was stated without proof as Theorem 7.2 of [5] in an appendix to that paper indicating how to pass from *localization* theorems in free homotopy to *completion* theorems. Because of this point of view, Theorem 7.2 of [5] employed the language and notation of p-profinite completion, for an arbitrary prime p. Of course in the context of our results we have $\hat{G} = \prod \hat{G}_{(p)}$, $\hat{X} = \prod \hat{X}_{(p)}$, where $\hat{G}_{(p)}$, $\hat{X}_{(p)}$ are the p-profinite completions of G, X, respectively.

The methods of proof of our purely algebraic results are themselves algebraic. That is, we do not use homotopy theory to obtain our results on finitely generated nilpotent groups; rather we go the other way, and use some of the ideas of the proof of our first generalization of Blackburn's theorem to prove our second, homotopy-theoretic, generalization, even though we do not here claim to have a homotopy-theoretic analog of our first generalization except when N is commutative.[2] In describing our methods of proof as algebraic, however, we by no means exclude the invocation of the *topology* of $\hat{G}$ as a compact Hausdorff group; indeed, we always regard $\hat{G}$ as a topological group in the natural way and exploit the fact that a basis at 1 in the natural topology of $\hat{G}$ consists of certain subgroups of $\hat{G}$ of finite index.

We are guided in our approach to the study of completion by previous experience in studying the localization of nilpotent groups and nilpotent actions. Indeed, we prove, systematically, many analogs of known results in localization theory (of course, we restrict ourselves to *finitely generated* nilpotent groups in order that completion be exact—otherwise the analogy would immediately break down). In particular, we use the technique in [7] for studying orbit-sets for nilpotent actions, but, in contrast to [7], we relate the technique here to completion. In view of the automatic nature of the *conceptual* transition from localization to completion—there is, of course, nothing automatic about *proving* the translated assertions!—we have not thought it necessary to repeat those details which the two theories have in common.

After a review in Section 1 of some basic notions relating to profinite completion, we devote Section 2 to discussing the theorem that completion is exact on the category of finitely generated nilpotent groups. This result is due to Bousfield and Kan [3], who proved a more general theorem;[3] we relate their exactness theorem to our result, which is crucial to our approach to Blackburn's theorem, that the terms of the lower central series of a finitely generated nilpotent group complete "properly," that is,

$$\widehat{\Gamma^i G} = \Gamma^i \hat{G}. \tag{0.1}$$

[2] Cf. [6], proof of Lemma 1.

[3] Schneebeli has also established a broader category on which completion is exact.

In Section 2 we also obtain the (continuous) action of $\hat{Q}$ on $\hat{N}$ compatible with the nilpotent action of Q on N, where Q, N are finitely generated nilpotent, and establish its uniqueness; and we obtain the generalization

$$\widehat{\Gamma_Q{}^i N} = \Gamma_{\hat{Q}}{}^i \hat{N} \tag{0.2}$$

of (0.1) needed for our proof of the generalizations of Blackburn's theorem. The last part of this section is concerned with our homotopy-theoretic generalization of Blackburn's theorem. The key to this generalization is Theorem 2.10.

In Section 3 we prove our algebraic generalization of Blackburn's theorem, basing ourselves on [7]. In a brief final section we speculate on the possibility of extending the generalizations of Blackburn's theorem to a wider class of groups and group-actions.

1. Preliminaries

We begin by recalling that, for an arbitrary group G, the *profinite completion* $\hat{G}$ of G is defined to be the inverse limit of the finite quotient groups of G; thus

$$\hat{G} = \varprojlim_{\alpha} G/N_\alpha , \tag{1.1}$$

where $\mathsf{N} = \{N_\alpha\}$ is the collection of normal subgroups of G of finite index. The group $\hat{G}$ inherits a natural topology as a subset of the Cartesian product $\prod_\alpha G/N_\alpha$ of discrete spaces and, in fact, becomes a compact, Hausdorff, totally disconnected topological group. We will always regard $\hat{G}$ as endowed with this topology; and we note that if $\mathsf{M} = \{M_\beta\}$ is a *cofinal* subcollection of N, that is, if for each α there exists β such that $M_\beta \subseteq N_\alpha$, then

$$\hat{G} = \varprojlim_{\beta} G/M_\beta , \tag{1.2}$$

as topological group. We then deduce the following elementary proposition which will be used in the sequel.

Proposition 1.1. *If G is a finitely generated nilpotent group, then* (i) *$\hat{G}$ is metrizable and* (ii) *there exists a descending sequence $\mathsf{M} = \{M_i\}$ of normal subgroups of G of finite index such that*

$$\hat{G} = \varprojlim_{i} G/M_i . \tag{1.3}$$

Proof. Since every subgroup of G is finitely generated, it follows that G has only countably many subgroups. This immediately implies that $\hat{G}$ is metrizable. Since the intersection of finitely many normal subgroups of finite index is again a normal subgroup of finite index, it is obvious that we can select a descending sequence $M_1 \supseteq M_2 \supseteq \cdots$ which is cofinal in N.

Returning to the general case, there is, plainly, a natural homomorphism $c\colon G \to \hat{G}$ whose image is dense in $\hat{G}$; moreover, since a finitely generated nilpotent group is residually finite, the next proposition follows immediately.

PROPOSITION 1.2. *If G is a finitely generated nilpotent group, then $c\colon G \to \hat{G}$ embeds G as a dense subgroup of $\hat{G}$.*

Note that the topology which G acquires as a subgroup of $\hat{G}$ is not the discrete topology; it is that topology in which a basis at the unity element consists of all the subgroups of G of finite index.[4] As for the topology of $\hat{G}$ itself we have the following result.

THEOREM 1.3. *If G is an arbitrary group, then the topology of $\hat{G}$ is such that a basis at 1 consists of certain subgroups of finite index.*

Proof. It is plain that, in the product topology for $\prod_\alpha G/N_\alpha$, a basis at 1 consists of certain subgroups S of $\prod_\alpha G/N_\alpha$ of finite index. Thus a basis at 1 for the topology of $\hat{G}$ consists of the subgroups $S \cap \hat{G}$ and these have finite index in $\hat{G}$.

It has been stated by Sullivan [8] that if G is finitely generated Abelian, then *every* subgroup of $\hat{G}$ of finite index is open; we give a proof for the sake of completeness. Thus let H be a subgroup of $\hat{G}$ of finite index. Then if $\exp \hat{G}/H = n$, we have (using additive notation) $n\hat{G} \subseteq H$ and certainly $n\hat{G}$ is of finite index in $\hat{G}$ (indeed, $\hat{G}/n\hat{G} \cong G/nG$) and hence in H. Thus H is a union of finitely many translates of the compact space $n\hat{G}$ and hence H is closed, and so open, in $\hat{G}$.

It may be conjectured that this assertion generalizes to finitely generated nilpotent groups. By essentially the same reasoning as above, it would be sufficient to prove the compactness of $\hat{G}^n$, the subgroup of $\hat{G}$ generated by the nth powers of elements of $\hat{G}$, which could perhaps be established with the aid of the commutator calculus. However, for the purposes of this paper, the weaker assertion of Theorem 1.3 will suffice.[5]

We next note that (1.1) and Proposition 1.2 immediately imply that if G is finitely generated nilpotent, then

$$\operatorname{nil} G = \operatorname{nil} \hat{G}. \tag{1.4}$$

Now let $\phi\colon G \to H$ be a homomorphism of groups. If H_0 is a normal subgroup of H of finite index, then $\phi^{-1}H_0$ is a normal subgroup of G of finite index. Thus ϕ

[4] In fact, $\hat{G}$ is precisely the "completion" of G in this topology.

[5] *Note added in proof.* R. B. Warfield, Jr., has proved this conjecture, basing himself on the notion of the p-adic completion of a nilpotent group. (See "Nilpotent Groups," Lecture Notes in Mathematics No. 513, Springer, New York, 1976, especially Theorem 7.10.)

induces $\hat{\phi}: \hat{G} \to \hat{H}$ which is a (continuous) homomorphism such that the diagram

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "c"'] & H \arrow[d, "c"] \\ \hat{G} \arrow[r, "\hat{\phi}"] & \hat{H} \end{tikzcd} \tag{1.5}$$

is commutative; moreover, $\hat{\phi}$ is uniquely determined by (1.5), since cG is dense in $\hat{G}$. It is plain that completion is functorial, and that c is a natural transformation of functors. Obviously, if 0 stands for the constant homomorphism,

$$\hat{0} = 0. \tag{1.6}$$

Further, we have

PROPOSITION 1.4. *If $\phi: G \to H$ is surjective, so is $\hat{\phi}: \hat{G} \to \hat{H}$.*

Proof. It follows from (1.5) that $\hat{\phi}\hat{G}$ is dense in $\hat{H}$, since $c\phi G = cH$ is dense in $\hat{H}$. But $\hat{\phi}\hat{G}$ is also closed in $\hat{H}$, so $\hat{\phi}\hat{G} = \hat{H}$.

We will not prove further exactness properties of completion here for the general case; but we will note in the next section that completion, in the general case, is right exact, and restricted to the category of finitely generated nilpotent groups, is exact.

However, we should remark that the conclusions of Propositions 1.1 and 1.2 by no means require that G be finitely generated nilpotent. As pointed out to us by H. R. Schneebeli, and as was noted some 30 years ago by M. Hall, for a given integer $k > 0$, a finitely generated group G has only finitely many subgroups of index k, so that G then has only countably many normal subgroups of finite index. Thus the conclusion of Proposition 1.1 holds for any finitely generated group G. Proposition 1.2 only requires that G be residually finite; this would be true if, for example, G were *virtually* finitely generated nilpotent, that is, if G admitted a finitely generated nilpotent normal subgroup of finite index. It seems likely that we could extend Theorem 3.1 to this class of groups.

2. The Exactness of Completion

We now place ourselves in the category of finitely generated nilpotent groups. Our first result depends on the exactness of completion in this category (Theorem 2.2) and will be generalized in Theorem 2.8 below.

PROPOSITION 2.1. *Let G be a finitely generated nilpotent group. Then $\widehat{\Gamma^k G} = \Gamma^k \hat{G}$ for all $k \geqslant 0$.*

Proof. It suffices to show (i) that $\Gamma^k G$ is dense in $\Gamma^k \hat{G}$ and (ii) and $\Gamma^k \hat{G}$ is closed in $\hat{G}$. For by (ii), $\Gamma^k \hat{G}$ is a compact Hausdorff group and it plainly inherits from $\hat{G}$ the property of having a basis at 1 consisting of subgroups of finite index. Thus we may apply Lemma 2.6 below to infer that the embedding $\Gamma^k G \subseteq \Gamma^k \hat{G}$ extends to a continuous homomorphism $\widehat{\Gamma^k G} \to \Gamma^k \hat{G}$ which is surjective by (i). Then Theorem 2.2 implies the conclusion. Since (i) is obvious, we concentrate on (ii) and argue by backward induction on k. For, since $\hat{G}$ is nilpotent, (ii) is certainly true for large k.

Now to prove (ii) it suffices to show that every sequence of elements of $\Gamma^k G$ which converges in $\hat{G}$ actually converges to an element of $\Gamma^k \hat{G}$; we assume inductively that this is known to be true with k replaced by $(k+1)$. There is a canonical surjection

$$G_{ab}^{k+1} \twoheadrightarrow \Gamma^k G / \Gamma^{k+1} G \tag{2.1}$$

of the $(k+1)$-fold tensor power of G_{ab} onto $\Gamma^k G/\Gamma^{k+1} G$. Thus if G is generated by $\{g_1, g_2, \ldots, g_r\}$, then every element a of $\Gamma^k G$ is expressible as

$$a = c_1 c_2 \cdots c_l b, \qquad l = r^k, \tag{2.2}$$

where $c_j = [g_{j_1}, [g_{j_2}, \ldots, [g_{j_k}, h_j] \ldots]]$, $h_j \in G$, $b \in \Gamma^{k+1} G$. Now let $a_n \in \Gamma^k G$, $n \geqslant 1$, and let $\{a_n\}$ converge to $a_0 \in \hat{G}$; we write, as in (2.2),

$$a_n = c_{1n} c_{2n} \cdots c_{ln} b_n,$$

where $c_{jn} = [g_{j_1}, [g_{j_2}, \ldots, [g_{j_k}, h_{jn}] \ldots]]$, $h_{jn} \in G$, $b_n \in \Gamma^{k+1} G$. By passing to subsequences if necessary, we may suppose that $\{h_{jn}\}$ converges to $h_{j0} \in \hat{G}$ and that $\{b_n\}$ converges to b_0 which, by our inductive hypothesis, belongs to $\Gamma^{k+1} \hat{G}$. Thus

$$a_0 = c_{10} c_{20} \cdots c_{l0} b_0,$$

where $c_{j0} = [g_{j_1}, [g_{j_2}, \ldots, [g_{j_k}, h_{j0}] \ldots]]$. It follows that $a_0 \in \Gamma^k \hat{G}$ and the inductive step is complete.

We now quote the following theorem of Bousfield and Kan [3].

THEOREM 2.2. *Let $H \rightarrowtail G \twoheadrightarrow K$ be a short exact sequence of finitely generated nilpotent groups. Then $\hat{H} \rightarrowtail \hat{G} \twoheadrightarrow \hat{K}$ is a short exact sequence of compact topological groups.*

We have used Theorem 2.2 to help establish Proposition 2.1. It might be of interest to point out that, conversely, the assertion of Proposition 2.1 may be used to derive Theorem 2.2.

To do this, we argue by induction on nil G. If nil $G = 1$, that is, if G (and hence also H, K) is commutative, then the conclusion follows from the structure theorem for finitely generated abelian groups, as observed by Sullivan [8]. If nil $G = d+1$ we set $\Gamma = \Gamma^d G$, $\Gamma' = \Gamma^d K$, and form the commutative diagram of short exact sequences

$$\begin{tikzcd}
H \cap \Gamma \arrow[r, tail] \arrow[d, tail] & H \arrow[r, two heads] \arrow[d, tail] & H/H \cap \Gamma \arrow[d, tail] \\
\Gamma \arrow[r, tail] \arrow[d, two heads] & G \arrow[r, two heads] \arrow[d, two heads] & G/\Gamma \arrow[d, two heads] \\
\Gamma' \arrow[r, tail] & K \arrow[r, two heads] & K/\Gamma'
\end{tikzcd} \tag{2.3}$$

Completing (2.3), we obtain

$$\begin{tikzcd}
\widehat{H \cap \Gamma} \arrow[r] \arrow[d] & \hat{H} \arrow[r, two heads] \arrow[d] & \widehat{H/H \cap \Gamma} \arrow[d] \\
\hat{\Gamma} \arrow[r] \arrow[d, two heads] & \hat{G} \arrow[r, two heads] \arrow[d, two heads] & \widehat{G/\Gamma} \arrow[d, two heads] \\
\hat{\Gamma}' \arrow[r] & \hat{K} \arrow[r, two heads] & \widehat{K/\Gamma'}
\end{tikzcd} \tag{2.4}$$

By our inductive hypothesis, the first and third columns of (2.4) are short exact; by Proposition 2.1 $\hat{\Gamma} \to \hat{G}$ and $\hat{\Gamma}' \to \hat{K}$ are injective. Also there are continuous homomorphisms $\widehat{G/\Gamma} \to \hat{G}/\hat{\Gamma}$ (by Lemma 2.6 below; since, by Theorem 1.3, $\hat{G}$ has a basis at 1 consisting of certain subgroups of finite index, so does $\hat{G}/\hat{\Gamma}$) and $\hat{G}/\hat{\Gamma} \to \widehat{G/\Gamma}$ (by (1.6)) such that the diagram

$$\begin{tikzcd}
& \hat{G} \arrow[dl] \arrow[dr] & \\
\widehat{G/\Gamma} \arrow[rr, shift left] & & \hat{G}/\hat{\Gamma} \arrow[ll, shift left]
\end{tikzcd}$$

commutes, showing that $\hat{G}/\hat{\Gamma} = \widehat{G/\Gamma}$. Thus the second (and third) rows of (2.4) are short exact. Virtually the same argument as that just given establishes the right exactness of completion in general; in particular, the second column of (2.4) is exact at $\hat{G}$ and the first row of (2.4) is exact at $\hat{H}$. An easy diagram chase now shows that $\hat{H} \to \hat{G}$ is injective.

We remark that since subgroups of nilpotent groups are subnormal, Theorem 2.2 implies that if H is a subgroup of the finitely generated nilpotent group G, then $\hat{H}$ is a subgroup of $\hat{G}$.

Our next theorem is a consequence of Theorem 2.2.

THEOREM 2.3. *Let Q be a finitely generated nilpotent group acting nilpotently on the finitely generated nilpotent group N. Then there is a unique nilpotent (continuous) action of $\hat{Q}$ on $\hat{N}$ compatible with the given action in the obvious sense that*

$$c(xa) = (cx)(ca), \qquad x \in Q, \quad a \in N. \tag{2.5}$$

Proof. We form the semidirect product G of N and Q. Then G is finitely generated nilpotent and we have a right-split short exact sequence

$$N \rightarrowtail G \underset{\twoheadrightarrow}{\overset{s}{\leftarrowtail}} Q. \tag{2.6}$$

We apply Theorem 2.2 to (2.6), obtaining the map of right-split short exact sequences

$$\begin{tikzcd} N \arrow[r, tail] \arrow[d, tail, "c"] & G \arrow[r, two heads, shift right] \arrow[d, tail, "c"] & Q \arrow[l, tail, shift right, "s"'] \arrow[d, tail, "c"] \\ \hat{N} \arrow[r, tail] & \hat{G} \arrow[r, two heads, shift right] & \hat{Q} \arrow[l, tail, shift right, "\hat{s}"'] \end{tikzcd} \tag{2.7}$$

The bottom row of (2.7) provides us with a continuous nilpotent action of $\hat{Q}$ on $\hat{N}$, compatible with the given action of Q on N in the sense that (2.5) holds. Thus we have set up a function

$$\lambda\colon A(Q, N) \to A(\hat{Q}, \hat{N})$$

from the set $A(Q, N)$ of nilpotent actions of Q on N to the set $A(\hat{Q}, \hat{N})$ of continuous nilpotent actions of $\hat{Q}$ on $\hat{N}$ and the remaining claim of our theorem is that λ is injective.

Now an action of Q on N induces, as in Section 1, a unique action of Q on $\hat{N}$ in the sense that

$$c(xa) = x(ca), \qquad x \in Q, \quad a \in N, \quad c\colon N \to \hat{N}. \tag{2.8}$$

Forming the associated semidirect products we obtain the commutative diagram

$$\begin{tikzcd} N \arrow[r, tail] \arrow[d, tail, "c"] & G \arrow[r, two heads, shift right] \arrow[d, tail, "c'"] & Q \arrow[l, tail, shift right, "s"'] \arrow[d, equal] \\ \hat{N} \arrow[r, tail] & H \arrow[r, two heads, shift right] & Q \arrow[l, tail, shift right, "s'"'] \end{tikzcd} \tag{2.9}$$

Moreover, all arrows are continuous if we give N, G, and Q the topologies in which a basis at the identity consists of the subgroups of finite index and H the product topology $\hat{N} \times Q$. Conversely, such a diagram (2.9) determines a continuous action of Q on $\hat{N}$ compatible with the given action of Q on N, and hence the unique action of Q on $\hat{N}$ so compatible.

Returning to (2.7) we may use $c: Q \rightarrowtail \hat{Q}$ to pull back the bottom row and thus obtain the commutative diagram

$$\begin{tikzcd}
N \arrow[r] \arrow[d, tail, "c"] & G \arrow[r, shift right, two heads] \arrow[d, tail, "c'"] & Q \arrow[l, shift right, tail] \arrow[d, equal] \\
\hat{N} \arrow[r] \arrow[d, equal] & H \arrow[r, shift right, two heads] \arrow[d, tail, "c''"] & Q, \arrow[l, shift right, tail] \arrow[d, tail, "c"] \\
\hat{N} \arrow[r] & \hat{G} \arrow[r, shift right, two heads] & \hat{Q} \arrow[l, shift right, tail]
\end{tikzcd} \qquad c''c' = c: G \rightarrowtail \hat{G}. \tag{2.10}$$

It follows from this and the preceding remarks that $c: N \to \hat{N}$ induces $c_*: A(Q, N) \to A(Q, \hat{N})$, that c_* is injective and that $c: Q \to \hat{Q}$ induces $c^*: A(\hat{Q}, \hat{N}) \to A(Q, \hat{N})$ such that $c^*\lambda = c_*$. Thus λ is injective.

Remarks. (i) The fact that, if Q acts nilpotently on N, the induced action of Q on $\hat{N}$ is also nilpotent is easily proved by a direct argument.

(ii) We may strengthen the relationship $c^*\lambda = c_*$ proved above. See Corollary 2.7.

PROPOSITION 2.4. *Let $H \rightarrowtail G \twoheadrightarrow K$ be a short exact sequence of finitely generated nilpotent groups and let $\bar{H} \rightarrowtail \bar{G} \twoheadrightarrow \bar{K}$ be a short exact sequence of compact Hausdorff groups such that there is a commutative diagram*

$$\begin{tikzcd}
H \arrow[r, tail] \arrow[d, "u"] & G \arrow[r, two heads] \arrow[d, "v"] & K \arrow[d, "w"] \\
\bar{H} \arrow[r, tail] & \bar{G} \arrow[r, two heads] & \bar{K}
\end{tikzcd} \tag{2.11}$$

Then

(i) *if v, w are completions, so is u;*

(ii) *if u, v are completions, so is w;*

(iii) *if u, w are completions, and if the topology of $\bar{G}$ is such that a basis at $1 \in \bar{G}$ consists of certain subgroups of finite index, then v is also completion.*

Proof. Since (i) and (ii) are easy to prove, we will be content to demonstrate (iii). We require two lemmas. Note that neither lemma requires Theorem 2.2.

LEMMA 2.5. *A compact Hausdorff group $\bar{G}$ is a profinite group (i.e., an inverse limit of finite groups) if and only if it admits a basis at 1 consisting of certain subgroups of finite index.*[6]

[6] Cf. L. S. Pontryagin, "Topological Groups," Sect. 43, Princeton Univ. Press, Princeton, N.J., 1939.

Proof. Certainly a profinite group has the stated topology, so we have only to prove the converse. Let $\{U_\alpha\}$ be a basis at $1 \in \bar{G}$ consisting of subgroups U_α of finite index; plainly we may assume the subgroups U_α to be normal. The projections $p_\alpha\colon \bar{G} \to \bar{G}/U_\alpha$ induce a continuous homomorphism $p\colon \bar{G} \to \varprojlim_\alpha \bar{G}/U_\alpha$ and we readily show that p is a topological isomorphism. First, p is injective since, $\bar{G}$ being Hausdorff, $\bigcap_\alpha U_\alpha = \{1\}$. Second, p is surjective since, just as in the discussion in Section 1, $p\bar{G}$ is dense in $\varprojlim_\alpha \bar{G}/U_\alpha$; but $p\bar{G}$ is compact and $\varprojlim_\alpha \bar{G}/U_\alpha$ is Hausdorff, so $p\bar{G}$ is closed in $\varprojlim_\alpha \bar{G}/U_\alpha$. We finally invoke the Hausdorff property of $\varprojlim_\alpha \bar{G}/U_\alpha$ again to infer that p is a homeomorphism.

LEMMA 2.6. *Let $v\colon G \to \bar{G}$ be a homomorphism from the finitely generated nilpotent group G to the compact Hausdorff group $\bar{G}$ topologized in such a way that a basis at* 1 *consists of certain subgroups of finite index. Then v has a unique extension to a (continuous) homomorphism $\phi\colon \hat{G} \to \bar{G}$.*[7]

Proof. We adopt the notation of Lemma 2.5 and write v_α for $p_\alpha v\colon G \to \bar{G}/U_\alpha$. Then if $U_\alpha \subseteq U_\beta$, there is an obvious commutative diagram

$$\begin{tikzcd} G \arrow[r, "v_\alpha"] \arrow[dr, "v_\beta"'] & \bar{G}/U_\alpha \arrow[d] \\ & \bar{G}/U_\beta . \end{tikzcd}$$

Passing to profinite completions, by functoriality, we get unique extensions $\hat{v}_\alpha\colon \hat{G} \dashrightarrow \bar{G}/U_\alpha$ of v_α, which render commutative the diagram

$$\begin{tikzcd} \hat{G} \arrow[r, "\hat{v}_\alpha"] \arrow[dr, "\hat{v}_\beta"'] & \bar{G}/U_\alpha \arrow[d] \\ & \bar{G}/U_\beta . \end{tikzcd}$$

The continuous homomorphisms $\hat{v}_\alpha$ then induce a continuous homomorphism $\hat{v}\colon \hat{G} \to \varprojlim_\alpha \bar{G}/U_\alpha$ such that

$$\hat{v}c = pv\colon G \to \varprojlim_\alpha \bar{G}/U_\alpha;$$

and this equation determines $\hat{v}$ uniquely since cG is dense in G. Finally we set $\phi = p^{-1}\hat{v}$ and the lemma is proved.

[7] More generally, if G, furnished with the topology described prior to Theorem 1.3, is a dense subgroup of a compact Hausdorff group G', then $v\colon G \to \bar{G}$ extends (uniquely) to $\varphi\colon G' \to \bar{G}$ by a uniform continuity argument; cf. J. L. Kelley, "General Topology," Chap. 6, Van Nostrand, Princeton, N.J., 1955.

We now revert to the proof of Proposition 2.4(iii). We have

$$\begin{tikzcd} H \arrow[r, tail, "\mu"] \arrow[d, tail, "c"] & G \arrow[r, two heads, "\epsilon"] \arrow[d, "v"] & K \arrow[d, tail, "c"] \\ \hat{H} \arrow[r, tail, "\nu"] & \bar{G} \arrow[r, two heads, "\eta"] & \hat{K} \end{tikzcd};$$

we use Lemma 2.6 to extend v to φ: $\hat{G} \to \bar{G}$. Thus we have the diagram (using Theorem 2.2)

$$\begin{tikzcd} H \arrow[r, tail, "\mu"] \arrow[d, tail, "c"] & G \arrow[r, two heads, "\epsilon"] \arrow[d, tail, "c"'] & K \arrow[d, tail, "c"'] \\ \hat{H} \arrow[r, tail, "\hat{\mu}"] \arrow[d, equal] & \hat{G} \arrow[r, two heads, "\hat{\epsilon}"] \arrow[d, "\varphi"'] & \hat{K} \arrow[d, equal] \\ \hat{H} \arrow[r, tail, "\nu"] & \bar{G} \arrow[r, two heads, "\eta"] & \hat{K} \end{tikzcd}$$

with $\varphi c = v$. Since $\varphi\hat{\mu}c = \varphi c\mu = v\mu = \nu c$, it follows, H being dense in $\hat{H}$, that $\varphi\hat{\mu} = \nu$. Similarly, $\eta\varphi = \hat{\epsilon}$. Thus φ is a (continuous) isomorphism and hence a homeomorphism and Proposition 2.4(iii) is proved.

COROLLARY 2.7. *Consider the commutative diagram*

$$\begin{tikzcd} & A(Q, N) \arrow[dl, tail, "\lambda"'] \arrow[dr, tail, "c_*"] & \\ A(\hat{Q}, \hat{N}) \arrow[rr, "c^*"] & & A(Q, \hat{N}) \end{tikzcd}$$

obtained in the proof of Theorem 2.3. *Then if* $c^*\omega = c_*\theta$, *it follows that* $\omega = \lambda\theta$, $\omega \in A(\hat{Q}, \hat{N})$, $\theta \in A(Q, N)$.

Proof. Let ω give rise to the right-split short exact sequence $\hat{N} \rightarrowtail H \rightleftarrows \hat{Q}$, and let θ give rise to the right-split short exact sequence $N \rightarrowtail G \rightleftarrows Q$. Then the equation $c^*\omega = c_*\theta$ implies the commutative diagram

$$\begin{tikzcd} \theta\colon & N \arrow[r, tail] \arrow[d, tail, "c"'] & G \arrow[r, shift right, two heads] \arrow[d, tail, "c'"] & Q \arrow[l, shift right, tail] \arrow[d, equal] \\ & \hat{N} \arrow[r, tail] \arrow[d, equal] & H' \arrow[r, shift right, two heads] \arrow[d, tail, "c''"] & Q \arrow[l, shift right, tail] \arrow[d, tail, "c"] \\ \omega\colon & \hat{N} \arrow[r, tail] & H \arrow[r, shift right, two heads] & \hat{Q} \arrow[l, shift right, tail] \end{tikzcd} \qquad (2.12)$$

But then Proposition 2.4 tells us that the bottom row of (2.12) arises by completing the top row, so that $\omega = \lambda\theta$.

Remark. In the corresponding *localization* theory for nilpotent actions on nilpotent groups [9; see especially Theorem 3.3], the function corresponding to c^* is bijective. In order to obtain such a result here we would certainly need to extend our completion theory beyond the category of finitely generated nilpotent groups.

Henceforth we regard the actions of Q on N and $\hat{Q}$ on $\hat{N}$ related by λ as *canonically associated.* We now enunciate our promised generalization of Proposition 2.1.

THEOREM 2.8. *Let the finitely generated nilpotent group Q act nilpotently on the finitely generated nilpotent group N. Then (see* [9])

$$\widehat{\Gamma_Q{}^k N} = \Gamma_{\hat{Q}}{}^k \hat{N}. \tag{2.13}$$

Proof. Let G be the semidirect product of of N and Q; then (see Theorem 2.3) $\hat{G}$ is the semidirect product of $\hat{N}$ and $\hat{Q}$. Moreover $\Gamma_Q{}^k N = \Gamma_G{}^k N$, $\Gamma_{\hat{Q}}{}^k \hat{N} = \Gamma_{\hat{G}}{}^k \hat{N}$. Thus we may assume in proving (2.13), that $N \lhd Q$, $\hat{N} \lhd \hat{Q}$. Further, it was shown in [9] that $\Gamma_Q{}^k N$ is then characterized by

$$\Gamma_Q{}^0 N = N, \qquad \Gamma_Q^{k+1} N = [Q, \Gamma_Q{}^k N], \qquad k \geqslant 0;$$

similarly for $\Gamma_{\hat{Q}}{}^k \hat{N}$. We may thus generalize the argument of Proposition 2.1 and need not enter into details. It should suffice simply to mention that (2.1) is replaced by its generalization, the canonical surjection

$$Q_{ab}^k \otimes N_{ab} \twoheadrightarrow \Gamma_Q{}^k N / \Gamma_Q^{k+1} N. \tag{2.14}$$

From this theorem we infer immediately the following generalization of (1.4).

COROLLARY 2.9. *Under the hypotheses of Theorem* 2.3,

$$\mathrm{nil}_Q N = \mathrm{nil}_{\hat{Q}} \hat{N}.$$

Proof. We have the inequalities

$$\mathrm{nil}_{\hat{Q}} \hat{N} \leqslant \mathrm{nil}_Q N \leqslant \mathrm{nil}_Q \hat{N} \leqslant \mathrm{nil}_{\hat{Q}} \hat{N}. \tag{2.15}$$

Here the first inequality follows from Theorem 2.8; the second from the embedding $c \colon N \rightarrowtail \hat{N}$; and the third from the embedding $c \colon Q \rightarrowtail \hat{Q}$. Thus all inequalities in (2.15) are, in fact, equalities.

We close this section by presenting an immediate and important topological consequence of the results of this section, which enables us to establish our homotopical generalization of Blackburn's theorem, as enunciated in the Introduction.

Given a nilpotent space X we may form the *canonical refinement* of its Postnikov tower (see the proof of Theorem II.2.9 of [10]). Thus if the Postnikov tower of X is $\cdots \to X_n \to^{p_n} X_{n-1} \to \cdots$ with the fiber of p_n being $K(\pi_n, n)$, where $\pi_n = \pi_n X$, then we obtain the canonical refinement by factoring p_n as the composite

$$X_n = Y_d \to \cdots \to Y_{i+1} \xrightarrow{q_i} Y_i \to \cdots \to Y_0 = X_{n-1}, \tag{2.16}$$

where $\mathrm{nil}_\pi \pi_n = d$, q_i is a principal fibration with fiber $K(G_i, n)$, with $G_i = \Gamma_\pi{}^i \pi_n / \Gamma_\pi^{i+1} \pi_n$, $\pi = \pi_1 X$. (With $n = 1$ we interpret $\Gamma_\pi{}^i \pi_1$ as $\Gamma^i \pi_1$).

THEOREM 2.10. *Let X be a nilpotent space of finite type. Then the canonical refinement of its Postnikov tower commutes with completion.*

Proof. We are asserting that if we complete (2.16), getting precisely

$$\hat{X}_n = \hat{Y}_d \to \cdots \to \hat{Y}_{i+1} \xrightarrow{\hat{q}_i} \hat{Y}_i \to \cdots \to \hat{Y}_0 = \hat{X}_{n-1}, \tag{2.17}$$

then (2.17) is the canonical refinement of $p_n : \hat{X}_n \to \hat{X}_{n-1}$; that is, $\hat{\pi}_n = \pi_n(\hat{X})$, $\hat{\pi} = \pi_1(\hat{X})$, $\mathrm{nil}_{\hat{\pi}} \hat{\pi}_n = d$, and $\hat{q}_i$ is a principal fibration with fiber $K(G_i^*, n)$, where $G_i^* = \Gamma_{\hat{\pi}}^i \hat{\pi}_n / \Gamma_{\hat{\pi}}^{i+1} \hat{\pi}_n$.

Now $\hat{\pi}_n = \pi_n \hat{X}$, $\hat{\pi} = \pi_1 \hat{X}$, by [4, Theorem 3.1] and $\mathrm{nil}_{\hat{\pi}} \hat{\pi}_n = d$ by Corollary 2.9. Since we may regard $Y_{i+1} \to^{q_i} Y_i \to K(G_i, n+1)$ as a fibration, it follows from the exactness of completion on finitely generated nilpotent groups (equivalently, from the fact that completion commutes with fibrations of nilpotent spaces of finite type) that

$$\hat{Y}_{i+1} \xrightarrow{\hat{q}_i} \hat{Y}_i \to K(\hat{G}_i, n+1)$$

is a fibration. But, by Theorem 2.8 (and Theorem 2.2), $\hat{G}_i = \Gamma_{\hat{\pi}}^i \hat{\pi}_n / \Gamma_{\hat{\pi}}^{i+1} \hat{\pi}_n = G_i^*$, so that the theorem is proved.

By this theorem we have entirely justified the claims made in the appendix to [5]; that is, the proofs of Theorems 7.1, 7.2, and 7.3 of that paper now consist of formal translations of the proofs given, in [5], of the corresponding statements in localization theory. Note that Theorem 7.2 of [5], asserting that if X is nilpotent of finite type and W homologically finite, then the map of free homotopy sets $(W, X) \to (W, \hat{X})$ induced by the completion $c : X \to \hat{X}$, is injective, is itself a homotopy-theoretic generalization of Blackburn's theorem, since, as remarked in the Introduction, we essentially recover Blackburn's theorem by taking $W = S^1$, $X = K(G, 1)$, for G finitely generated nilpotent.

3. The Generalized Blackburn Theorem

We again assume that Q is a finitely generated nilpotent group acting nilpotently on the finitely generated nilpotent group N. There is then an associated nilpotent action of $\hat{Q}$ on $\hat{N}$. Let a, b be elements of N. We prove:

THEOREM 3.1. *The following statements are equivalent:*

(A) *$a, b \in N$ are in the same Q-orbit;*

(B) *$a, b \in N$ are in the same $\hat{Q}$-orbit;*

(C) *for every normal Q-subgroup N_α of N of finite index, the images a_α, b_α of a, b in N/N_α are in the same Q-orbit.*

Before proving this theorem, we observe that the equivalence of (A) and (C) does indeed generalize Blackburn's theorem. For if $Q = N$, acting on itself by conjugation, then $a, b \in Q$ are in the same Q-orbit if and only if they are conjugate; every normal subgroup of Q is a Q-subgroup; and a_α, $b_\alpha \in Q/Q_\alpha$ are in the same Q-orbit if and only if they are conjugate.

Proof of Theorem 3.1. Plainly (A) $\Rightarrow$ (B). To prove that (B) $\Rightarrow$ (A) we proceed exactly as in the proof of Theorem 3.1 of [7]. Thus if $\mathrm{nil}_Q N = d + 1$, we set $\Gamma = \Gamma_Q{}^d N$, $\hat{\Gamma} = \Gamma_{\hat{Q}}{}^d \hat{N}$, and we have a map of central extensions

$$\begin{tikzcd} \Gamma \arrow[r, tail] \arrow[d, tail, "c"'] & N \arrow[r, two heads, "\kappa"] \arrow[d, tail, "c"'] & M \arrow[d, tail, "c"'] \\ \hat{\Gamma} \arrow[r, tail] & \hat{N} \arrow[r, two heads, "\hat{\kappa}"] & \hat{M} \end{tikzcd} \tag{3.1}$$

compatible with $c: Q \rightarrowtail \hat{Q}$; this is guaranteed by Theorem 2.8 and Proposition 2.4(ii). Here Q acts trivially on Γ and $\hat{Q}$ acts trivially on $\hat{\Gamma} = \Gamma_{\hat{Q}}{}^d \hat{N}$. Then, for each $a \in N$, (3.1) gives rise to the map of exact sequences

$$\begin{tikzcd} Q(a) \arrow[r, tail] \arrow[d, tail] & Q(\kappa a) \arrow[r] \arrow[d, tail] & \Gamma \arrow[r] \arrow[d, tail, "c"] & N \mid Q \arrow[r, two heads] \arrow[d] & M \mid Q \arrow[d] \\ \hat{Q}(ca) \arrow[r, tail] & \hat{Q}(\hat{\kappa} ca) \arrow[r] & \hat{\Gamma} \arrow[r] & \hat{N} \mid \hat{Q} \arrow[r, two heads] & \hat{M} \mid \hat{Q} \end{tikzcd} \tag{3.2}$$

where $Q(a)$ is the subgroup of Q consisting of elements $x \in Q$ leaving a fixed, $xa = a$; $N \mid Q$ is the orbit set for N under the Q-action; the other notations of (3.2) are self-evident; and the morphisms of (3.2) are described in [7]. We refer to [7] for the details of the argument, by induction on d, which enables us to deduce, first, that $Q(a) \rightarrowtail \hat{Q}(ca)$ and $Q(\kappa a) \rightarrowtail \hat{Q}(\hat{\kappa} ca)$ are completion maps (we require Theorem 2.2) and then, second, that $N \mid Q \to \hat{N} \mid \hat{Q}$ is injective. Of course, this latter assertion is precisely equivalent to our claim that (B) $\Rightarrow$ (A).

The implication (A) $\Rightarrow$ (C) is evident, so that it only remains to prove that (C) $\Rightarrow$ (B). To achieve this we need certain auxiliary results which we enunciate explicitly as they seem to have some independent interest.

PROPOSITION 3.2. *Let N be a finitely generated nilpotent group, let Q act on N, and let M be a subgroup of N of finite index. Then $\bigcap_{x\in Q} xM$ is also of finite index in N.*

Proof. Since N is finitely generated nilpotent, it follows that a subgroup H of N is of finite index if and only if there exists a positive integer k such that $a^k \in H$ for all $a \in N$. Thus there exists k such that $a^k \in M$ for all $a \in N$. If $x \in Q$, then $x^{-1}a^k = (x^{-1}a)^k \in M$ for all $a \in N$, so that $a^k \in xM$, $a^k \in \bigcap_{x\in Q} xM$ for all $a \in N$, and hence $\bigcap_x xM$ is of finite index in N.

Since $\bigcap_x xM$ is normal in N if M is normal in N, we immediately infer from Propositions 1.1 and 3.2.

COROLLARY 3.3. *Let N be a finitely generated nilpotent group and let Q act on N. Then there exists a descending sequence $\mathsf{N}* = \{N_i\}$ of normal Q-subgroups of N of finite index such that*

$$\hat{N} = \varprojlim_i N/N_i .$$

Now let Q be a finitely generated nilpotent group and let $\{P_i\}$ be the (countable) collection of its normal subgroups of finite index. Taking the sequence $\mathsf{N}*$ of Corollary 3.3, let R_i be maximal in Q with the property of acting trivially on N/N_i. Then R_i is normal in Q and Q/R_i acts faithfully on N/N_i. It follows that Q/R_i is finite so that R_i is of finite index in Q. Now set $Q_i = P_1 \cap P_2 \cap \cdots \cap P_i \cap R_i$. Then Q_i is a normal subgroup of Q of finite index, Q/Q_i acts on N/N_i, and $\{Q_i\}$ is a cofinal descending sequence in $\{P_i\}$, so that

$$\hat{Q} = \varprojlim_i Q/Q_i .$$

We have thus proved

PROPOSITION 3.4. *If Q is a finitely generated nilpotent group acting on the finitely generated nilpotent group N, then we may find a descending sequence $\{N_i\}$ of normal Q-subgroups of N of finite index and a descending sequence $\{Q_i\}$ of normal subgroups of Q of finite index, such that Q_i acts trivially on N/N_i and*

$$\hat{N} = \varprojlim_i N/N_i , \qquad \hat{Q} = \varprojlim_i Q/Q_i .$$

We now return to the proof of Theorem 3.1, that is, of the implication (C) $\Rightarrow$ (B). We adopt the data and notation of Proposition 3.4. Then hypothesis (C) clearly implies that, for each i, there are elements $\xi_i \in Q/Q_i$ such that $\xi_i a_i = b_i$, where a_i, b_i are the images of a, b in N/N_i. It remains to show that we may choose the elements ξ_i such that $\xi_{i+1} \mapsto \xi_i$ under $Q/Q_{i+1} \twoheadrightarrow Q/Q_i$. For then the elements ξ_i will determine an element $\xi \in \hat{Q} = \varprojlim_i Q/Q_i$, such that $\xi a = b$ and we will have concluded the truth of (B). Since the assertion is evident if Q is finite, we henceforth assume Q is infinite. Consider the (finite) set of elements $\xi_{1k} \in Q/Q_1$ such that $\xi_{1k}a_1 = b_1$. For each i, k consider the (finite) set S_{ik} of elements $\xi_{ik} \in Q/Q_i$ such that $\xi_{ik}a_i = b_i$ and ξ_{ik} projects to ξ_{1k}. Let $S_k = \bigcup_i S_{ik}$, $S = \bigcup_k S_k$. Then S is an infinite set, so some S_k is infinite. Choose such a k and write ξ_1 for ξ_{1k}. There are then infinitely many values of i such that there exists $\xi_{ik} \in S_{ik}$, from which it immediately follows that there exists $\xi_{ik} \in S_{ik}$ for *every* i. We now begin again by considering the (finite) set of elements $\xi_{2k} \in Q/Q_2$ such that $\xi_{2k}a_2 = b_2$ and ξ_{2k} projects to $\xi_1 \in Q/Q_1$. As before we find at least one such element $\xi_2 \in Q/Q_2$ such that there exists, for each $i \geqslant 2$, $\xi_{ik} \in Q/Q_i$ projecting to ξ_2 and such that $\xi_{ik}a_i = b_i$. Proceeding in this way, we build up the required element $\xi \in \hat{Q}$ and the proof is complete.

Remarks. (i) The argument in the last paragraph is analogous to that used in proving Lemma 2 in [11].

(ii) No delicacy is required in the choice of sequence $\{Q_i\}$ in the special case of Blackburn's theorem, that is, when $Q = N$, acting on itself by conjugation. For then we simply take $Q_i = N_i$.

4. Possible Generalizations

Since the appearance of Blackburn's paper [2], there has been continued interest on the part of group theorists in enlarging the class of groups for which the conclusion of his theorem is valid. For example, we mention [12], where one may also find references to earlier contributions by other workers. Recall that a group G is *polycyclic* if it admits a subnormal series $G = G_0 \rhd G_1 \rhd \cdots \rhd G_n = \{1\}$ with each G_i/G_{i+1} cyclic, and that G is *virtually polycyclic* if it admits a subgroup of finite index which is polycyclic. The main result of [12] is then the following generalization of Blackburn's theorem.

Theorem 4.1 (Formanek, Remeslennikov). *If G is a virtually polycyclic group and $x, y \in G$, then x and y lie in the same conjugacy class if and only if the images of x and y in every finite quotient group of G lie in the same conjugacy class.*

By analogy with the group-theoretic situation, we may define, for a given group π, a *polycyclic* π-module M by requiring that there be a finite descending chain $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n = \{0\}$ of π-submodules such that each M_i/M_{i+1} is cyclic (as an Abelian group). We may further define M to be a *virtually polycyclic* π-module by requiring π to admit a subgroup π' of finite index such that the induced π'-module structure on M is polycyclic. These notions lead, in an evident way, to the notions of polycyclic space and virtually polycyclic space. Thus, a connected CW-complex is (virtually) polycyclic if (i) $\pi_1 X$ is a (virtually) polycyclic group, (ii) each $\pi_n X$, $n \geqslant 2$, viewed as a $\pi_1 X$-module, is (virtually) polycyclic. (The closely related notion of virtually nilpotent space has already been defined and investigated in [13].)

More generally we may describe the action of a group Q on a group N *as polycyclic* if N admits a finite descending subnormal series of Q-subgroups $N = N_0 \supseteq N_1 \supseteq \cdots \supseteq N_n = \{0\}$ such that N_i/N_{i+1} is cyclic; and the action of Q on N is *virtually polycyclic* if there exists a subgroup Q' of Q of finite index such that the action restricted to Q' is polycyclic. Then Theorem 4.1, together with Theorem 3.1 and with Theorem 7.2 of [5], suggest the following questions.

Question 1. Let Q be a virtually polycyclic group with a virtually polycyclic action on the group N and let $a, b \in N$. If, for every normal Q-subgroup N_α of N of finite index the images a_α, b_α of a, b in N/N_α are in the same Q-orbit, does it follow that a, b are in the same Q-orbit?

Question 2. If X is a virtually polycyclic space and W is a homologically finite CW-complex, is the map $(W, X) \to (W, \hat{X})$ of free homotopy sets, induced by the completion $c\colon X \to \hat{X}$, injective?

We remark that a virtually polycyclic space has "good" homotopy groups in the sense of [4; Theorem 3.1 *et seq.*]. Indeed, Sullivan speaks of groups commensurable with solvable groups of finite type[8] and these are precisely the virtually polycyclic groups. It therefore appears reasonable to study Question 2 above for this class of spaces X.

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[8] By finite type, we mean, following Sullivan, that every subgroup is finitely generated.

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